# Two-Bit Adaptive LMS-Like Filtering

Amin Z. Sadik and Zahir M. Hussain
School of Electrical and Computer Engineering, RMIT University
Melbourne, Victoria 3000, Australia
amin.sadik@rmit.edu.au; zmhussain@ieee.org

## Abstract

*The conventional LMS family of adaptive algorithms fail to converge if translated to short-word-length (single-bit, two-bit, or ternary) domains. As such the distinctive advantage of short-word-length systems, namely, the hardware implementation simplicity, has not been put into effect. We propose a 2-bit-domain LMS adaptive filtering structure for noise cancelling where all input, output, and filter coefficients are in 2-bit format. Simulation results showed that the proposed adaptive structure exhibits performance that is equivalent to the infinite-precision LMS algorithm.*

## 1. Introduction

The conventional infinite-precision LMS adaptive approach has proved to be efficient in finding optimal minimum mean-square solution for a wide variety of linear estimation problems. That approach is based on the recursive application of the steepest descent principle to direct the weight-vector towards the optimal solution predicted by Wiener-Hopf equations [1]-[3].

Unfortunately, when the LMS adaptive algorithm is used in short-word-length system (single-bit, two-bit or ternary), the effect of the harsh quantization process (which is a nonlinear process) seems to have a severe impact on the operation of the standard LMS algorithm and its variations such that their performance collapses and accordingly fails to converge to the Wiener solution. Short-Word-Length systems enjoy very attractive properties as compared to their multi-bit counterparts. The short-word-length implementation produces relatively higher performance and lower hardware complexity, however, their useability in practice (e.g., in communication systems) is very limited due to its unresolved adaptivity problem. In fact, there has been no attempt towards the topic of short-word-length adaptive filtering.

A key application of adaptive systems is noise cancelling. For estimation of narrow-band signals with unknown frequency content, the adaptive realization of Wiener filter (using Widrow-Hoff LMS algorithm) can be utilized with the desired (reference) signal $d(k)$ chosen to be the noisy input sequence $x(n) = s(n) + \nu(n)$ itself, while a delayed version of $x(n)$ is chosen as the filter input.

In this paper, an all-2-bit adaptive LMS-like filtering is introduced, analyzed, and simulated. The input (noisy) signal, the output (estimated) signal, and the FIR filter coefficients are all assumed to be in 2-bit format; as such the system would be simple to implement using FPGA technology. Moreover, the need for decimators, interpolators, and multibit multipliers is eliminated. The proposed adaptive structure is shown to converge in the LMS sense. As the algorithm processes blocks of input data, it will be called 2BLL (2-Bit Block LMS-Like) to highlight the similarities with the standard LMS.

## 2 A 2-bit Adaptive Approach

### 2.1 Gradient Approximation

The LMS algorithm is an application of the steepest descent method, that is, the tap weight vector $\mathbf{w}$ is iteratively updated. According to this method, the instantaneous gradient is equal to $-2e(k)\mathbf{x}(k)$. Thus, the well-known Widrow-Hoff LMS algorithm [2] is introduced:

$$\mathbf{w}(k+1) = \mathbf{w}(k) - 2\mu e(k)\mathbf{x}(k). \quad (1)$$

If we apply the same analysis used in the standard LMS algorithm to a 2-bit system (with both the input and the output in 2-bit format, i.e., $\mathbf{x}(k)$ and $\mathbf{y}(k) \in \{\pm 1, \pm 1/2\}$) would definitely drive it to divergence due to the the harsh quantization of the input, which will produce a switching instantaneous gradient function $(\partial e^2(k)/\partial w_i)$ that jumps around large quantities; a situation that cannot be tolerated by the adaptive algorithm.

It is necessary to suggest a feasible solution which must manifest the principle of minimum disturbance which has already been utilized by the existing adaptive algorithms.

In addition to the approximation made by Widrow (i.e., $E[e^2(k)] \rightarrow e^2(k)$), it should be taken into account that the error is no longer a continuous variable, in fact $e(k) \in \{\pm 1, 0, \pm 3/2, \pm 2\}$, hence the effect of $\partial e(k)/\partial w_i$ would be replaced by $\gamma(k) \in \pm 1, 0, \pm 3/2, \pm 2\}$, noting that $w_i \in \{\pm 1, \pm 1/2\}$.

To comply with the steepest descent method, the partial differentiation equivalent function $\gamma(k)$ should undergo a minimum change during successive iterations. Thus, the only non-zero choice is to use the approximation $\gamma(k) = 1$. The Short-Word-Length gradient function will be given as

$$\bigtriangledown(k) = -2e(k). \quad (2)$$

The above formula will be utilized to reach an approximation to the optimal Wiener solution **w*** in 2-bit-domain adaptive filter as will be seen in the next Subsection.

## 2.2. System Design

The proposed adaptive filtering structure is to be entirely achieved in 2-bit domain and thus has to operate at an oversampled rate. Intuitively, due to the harsh quantization in the 2-bit domain, one might exclude the sample-by-sample scheme of operation as per the standard LMS algorithm. This suggests using blocks of samples instead (the LMS algorithm can be viewed as a special case of the block-LMS with block-length = 1, [12]).

The input signal is assumed to be a 2-bit sigma-delta modulated gaussian noise corrupted sinusoid. Let the 2-bit input signal vector at time index $n$ is expressed as

$$\mathbf{x}(n) = [x(n), x(n-1), \cdots, x(n-M+1)]^T \quad (3)$$

where $[.]^T$ indicates transposition, $L$ denotes the block length ($1/L$ represents the updating rate), and $M$ represents the length of the interpolated 2-bit FIRb filter (i.e., $M = m \times$ OSR, where $m$ is the multi-bit Nyquist rate equivalent filter order). As per conventional BLMS, one may assume any of the following cases: $L < M$, $L = M$, or $L > M$. However, the first two cases ($L \leq M$) are probably preferred in most applications [12]. In addition, for a simpler implementation, it is better off to choose power-of-2 values for $M$ and $L$.

Let the 2-bit coefficients vector of the FIRb filter at time index $n$ be denoted by

$$\mathbf{w}(n) = [w_o(n), w_1(n), \cdots, w_{M-1}(n-M+1)]^T \quad (4)$$

and the 2-bit estimation output vector at time $n$ be denoted as

$$\mathbf{y}(n) = [y(n), y(n-1), \cdots, y(n-M+1)]^T. \quad (5)$$

To proceed in terms of block notation, let $k$ refer to the block index which is related to the original sampling index $n$ as follows

$$n = kL + i \bmod L, \qquad k = 1, 2, 3, ... \quad (6)$$

The 2-bit input data for block $k$ is therefore defined by the set $\{x(kL+i)\}_{i=0}^{i=L-1}$, which can be expressed in matrix form as

$$\mathbf{A}(k) = [\mathbf{x}(kL), \mathbf{x}(kL+1), \cdots, \mathbf{x}(kL+L-1)]. \quad (7)$$

Over this block of input data, the tap-weight vector of the filter is held constant at the value $\mathbf{w}(k)$.

The estimation output, $\hat{x}(kL+i)$, produced by the FIRb filter in response to the input signal vector $\mathbf{x}(kL+i)$ is given by

$$\hat{x}(kL+i) = \mathbf{w}^T(k)\ \mathbf{x}(kL+i). \quad (8)$$

This signal is in multi-bit format and should be remodulated into 2-bit representation. Practically, this can be done by introducing a sigma-delta modulation stage. This $\Sigma\Delta$ modulator must have a flat signal frequency response over the bandwidth of interest [5]. This implies that the $\Sigma\Delta$ modulator should not modify the specifications of the estimation signal, moreover, it requires operation at an oversampled rate (OSR). This requirement will be satisfied as the input signal has already been assumed here to be a $\Sigma\Delta$ modulated bit-stream. The 2-bit version of the estimation output $y(kL+i)$ can be then described as

$$y(kL+i) = \mathrm{sgn}[\alpha \hat{x}(kL+i)]. \quad (9)$$

where $\alpha$ is a gain parameter. Using the well-known linear approximation to model the behavior of the sigma-delta modulator [7], the output can thus be given as

$$y(kL+i) = \alpha \hat{x}(kL+i) + Q_y(kL+i) \quad (10)$$

where $Q_y(kL+i)$ represents the (shaped) quantization noise due to the modulation effect; given by the following convolution

$$Q_y(kL+i) = \alpha \sum_{j=0}^{M-1} h_j q(kL+i-j). \quad (11)$$

where $h_j$ characterizes the impulse response coefficients of the noise transfer function of the sigma-delta modulator (note that the term $h_o$ is always unity) and $q(kL+i)$ is the quantization noise. Assuming $q(kL+i)$ is an i.i.d random process and also independent of the input signal $\hat{x}(kL+i)$. The quantization noise vector is defined as

$$\mathbf{Q}_y(k) = [Q_y(kL), Q_y(kL+1), \cdots, Q_y(kL+M-1)]. \quad (12)$$

Substituting (8) into (10), the 2-bit output is

$$y(kL+i) = \alpha \mathbf{w}^T(k)\, \mathbf{x}(kL+i) + Q_y(kL+i). \quad (13)$$

or, in matrix form,

$$\mathbf{y}(k) = \alpha \mathbf{A}^T(k)\, \mathbf{w}(k) + \mathbf{Q}_y(k), \quad (14)$$

where $\mathbf{A}(k)$ is an $M \times L$ matrix defined in (7).

To develop an adjustment formula for the tap weights vector, we start with defining the error signal. Recalling (2) and taking into consideration the block nature of operation of the proposed 2BLL, the error vector at any time instant is given by

$$\mathbf{e}(kL+i) = \mathbf{x}(kL+i) - \mathbf{y}(kL+i), \quad (15)$$

and is defined at block $k$ as

$$\mathbf{e}(k) = \mathbf{x}(kL) - \mathbf{y}(kL). \quad (16)$$

The tap-weights should be restricted to 2-bit format, i.e., $w_j \in \{-1, +1\}$. This task can be carried out by using a 2-bit quantizer. Thus, the updating formula may be described as

$$\mathbf{w}(k+1) = \text{sgn}[\mathbf{w}(k) + \mu \mathbf{e}(k)] \quad (17)$$

where $\mu$ is a proposed step-size parameter of the 2BLL filtering. Again, using the linear approach to model the quantizer, the quantizer output cab be represented as a combination of the quantizer input and a white quantization noise. Thus, by substituting (15) into (17), the updating formula can be approximated as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu[\mathbf{x}(k) - \mathbf{y}(k)] + \mathbf{Q}_w(k). \quad (18)$$

where $\mathbf{Q}_w(k)$ is an $M$-by-1 vector which represents the tap-weight quantization noise. Now, substituting (14) into (18) yields the final updating formula

$$\begin{aligned} \mathbf{w}(k+1) \quad = \quad & \mathbf{w}(k) + \mu[\mathbf{x}(k) - \\ & \alpha \mathbf{A}^T(k)\mathbf{w}(k) + \mathbf{Q}_y(k)] + \mathbf{Q}_w(k). \quad (19) \end{aligned}$$

To this end, (19)can be utilized to construct the proposed 2BLL adaptive structure which is illustrated in Fig.(1).

It is convenient here to recall the updating equation of the conventional BLMS using the same notation as above:

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \sum_{i=0}^{M-1} \mathbf{x}(kL+i) e(kL+i). \quad (20)$$

The second term of the right-hand side of (20) represents the block gradient, which is a linear correlation between the error signal and the input vector. That is, the error signal is a single sample (produced by averaging the most recent $L$ error samples). On the other hand, according to (17), the block gradient in 2BLL is represented by a 2-bit quantized error vector which is described in (19) as $\mathbf{x}(k) - \alpha \mathbf{A}^T(k)\mathbf{w}(k) + \mathbf{Q}_y(k)$.

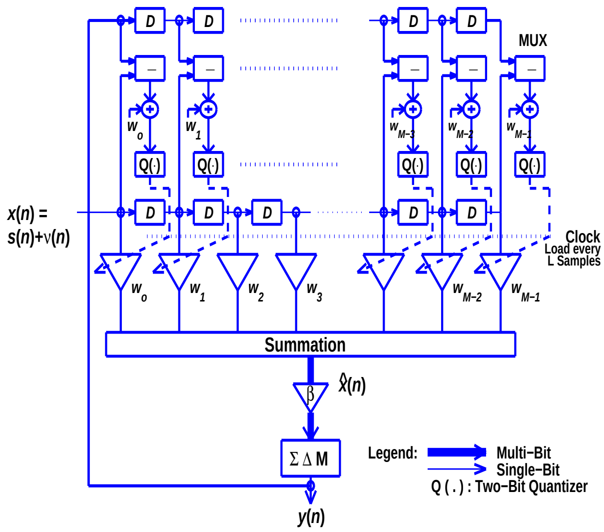


**Figure 1. A Proposed block-diagram of a 2-bit LMS-like adaptive filtering.**

## 3 Stability of 2BLL

The convergence properties of the 2BLL filtering are the required bounds on the convergence constant ($\mu$), rate of convergence, and the misadjustment. These properties are equivalent to that of the standard multi-bit BLMS and can be found in several works (e.g., [13][12]). However, it is expected that the convergence accuracy (misadjustment) in the 2BLL case would be relatively more noisy because of the crude approximation adopted in the iterative calculation of the gradient function. In addition, convergence of the 2BLL is affected by two additional issues: the intolerable harsh quantization errors imposed on the recursive weight updating formula (see (19)), and the optimum input dynamic range of the SD modulator stage in the sense of maximum attainable SNR.

### 3.1 Dynamic Range of the SD Modulator

The input to the SD modulator stage $\hat{x}(k)$ is the convolution between the input $x(k)$ and the FIRb filter coefficients (both are interpolated by a factor of OSR). The gain parameter $\alpha$ is introduced to ensure the stability of the SD modulator and is chosen such that it provides maximum SNR. This depends on the SD design parameters as well as the block length ($L$), that is,

$$\alpha = 1/L. \quad (21)$$

This may suggest that the performance of the 2BLL would be improved further in terms of SNR when a suitable adaptive SD modulator scheme is utilized. Adaptive

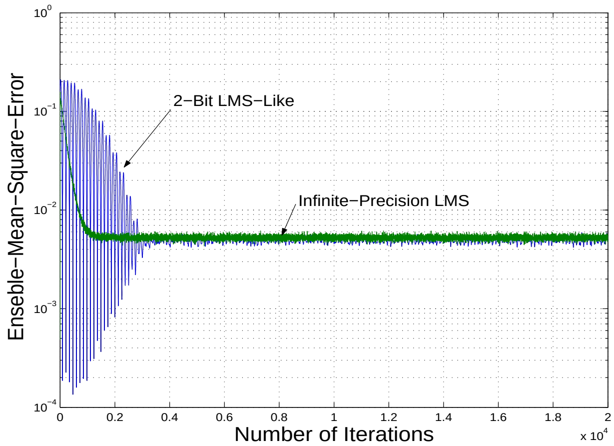


**Figure 2. A comparison between Learning curve of the 2BLL (oversampled) and the infinite-precision LMS algorithm with equal filter order ($m = 20$) and $\mathrm{SNR_i} = 14.2$ dB (input noise power= -23.2 dB) .**

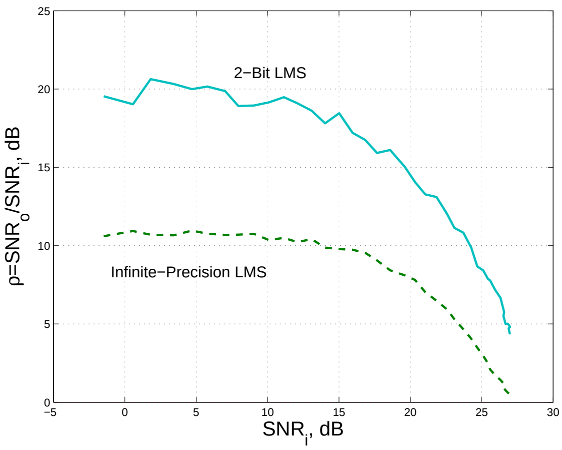


**Figure 3. The improvement in SNR ($\rho$) in the 2BLL adaptive filter compared to that of the standard infinite-precision LMS algorithm.**

dynamic range single-bit SD modulators can be found in several works [8]-[9].

### 3.2 The Updating Step-Size

The effective step-size $\hat{\mu}$ in the updating equation given in (19) can be expressed as

$$\hat{\mu} = \alpha\, \mu = \mu/L. \tag{22}$$

This expression conforms to that utilized in the conventional BLMS. A necessary (but not sufficient) stability bound on the step-size parameter $\mu$ of the LMS filter for large FIR length $(M)$ is given in [10] as follows

$$\hat{\mu} = 2/(MS_{\max}), \quad \text{or} \quad \mu = 2/(mS_{\max}) \tag{23}$$

where $S_{\max}$ is the maximum value of the power spectral density $S(\Omega)$. According to simulation results, the proposed adaptive filter is stable for $0 < \mu \leq 0.25$.

## 4 Simulation and Discussion

The 2BLL performance has been verified using MATLAB simulation. The input signal $x(n)$ is assumed, throughout the simulation unless otherwise is stated, to be the 2-bit (oversampled) digitized version of the original sinusoid $s(t)$ which is distorted by additive white Gaussian noise $\nu(t) \in \mathcal{N}(\sigma^2, 0)$. The sinusoid has an amplitude $A = 0.5$ and a frequency $f_o = 2000$ Hz. The Nyquist rate FIR filter order is assumed as $m$=20, and the oversampling ratio is chosen as $OSR = 128$, therefore, the number of 2-bit coefficients is $M = 2560$.

### 4.1 Learning Curves

To evaluate the convergence properties of the 2-bit adaptive filtering 2BLL, Fig.(2) depicts a comparison between learning curves of the oversampled 2BLL ($M = 20\times$ OSR, where OSR=128) and the conventional infinite-precision LMS algorithms ($m$=20).

### 4.2 Signal-to-Noise Ratio (SNR)

This paper is concerned with noise cancellation from narrow-band 2-bit input signals. In order to assess the improvement in the output SNR ($\mathrm{SNR}_o$) as a function of the SNR at the input ($\mathrm{SNR}_i$), a performance parameter $\rho$ is defined as $\rho = \mathrm{SNR}_o(\mathrm{dB})/\mathrm{SNR}_i(\mathrm{dB})$. It is noteworthy here that these SNR terms refer to *in-band* signal-to-noise ratios, as we are not interested in frequency bands outside it. Fig.(3) shows a performance comparison (in terms of $\rho$) between the 2BLL and the infinite-precision LMS. It is clear that the 2BLL outperforms LMS for $\mathrm{SNR}_i > 16$ dB. According to (19), this phenomenon would be attributed to dithering effects, as the low $\mathrm{SNR}_i$ would be compensated by the uncorrelated white noise due to the harsh quantization. On the other hand, the converse occurs for $\mathrm{SNR}_i < 16$. This is expected, using the same argument. However, both algorithms deteriorate (i.e., $\rho < 0$ dB) at almost the same value of $\mathrm{SNR}_i$.

To emphasize the noise cancelling effect of the 2BLL on the original noisy sinusoid analog input signal (before SD

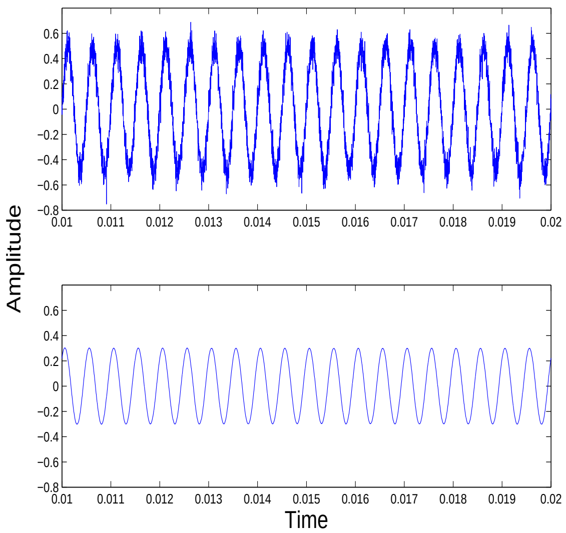


**Figure 4. A comparison between the analog noisy sinusoid input (SNR=10) and the corresponding steady-state estimated signal.**

modulation), Fig.(4) shows this input with $f_o$=2 kHz and SNR=10 dB (noise power=-19 dB), along with the demodulated estimation signal (i.e., the decimated and low-pass-filtered version of the estimation signal).

On the other hand, $\rho$ is affected by the oversampling ratio (OSR) which is a decisive design parameter in 2-bit systems. As increasing OSR improves correlation, on which this de-noising is based, $\rho$ is improved by increasing OSR. However, this will increase the FIRb length $M$, and therefore, selecting an appropriate OSR becomes a matter of compromise between hardware implementation simplicity and design requirements.

## 5 Conclusions

The lack of adaptive LMS structures in Short-Word-Length systems has substantially limited their useability despite their major advantage of hardware simplicity over multi-bit systems. In this paper we tackled this problem by introducing a 2-bit block LMS-like algorithm that seems to be quite promising. It has been shown that the proposed 2-bit algorithm is comparable in performance to the multi-bit conventional LMS algorithm. We expect this approach to open the door for short-word-length systems to be a practical alternative to multi-bit signal processing systems.

## 6 Acknowledgement

This work is supported by the Australian Research Council under the ARC Discovery Grant DP0557429.

## References

[1] B. Widrow, "Thinking about thinking: the discovery of the LMS algorithm," *IEEE Signal Processing Magazine*, pp. 100-106, Jan. 2005.

[2] B. Widrow *et al*, "Stationary and non-stationary learning chracterstics of the LMS adaptive filter," *Proceedings of IEEE*, vol. 64, no. 8, pp. 1151-1162, Aug. 1976.

[3] B. Widrow and M. Kamenetsky, "On the statistical efficiency of the LMS family of adaptive algorithms," *Proceedings of IEEE Conference on Neural Networks*, vol. 4, pp. 2872-2880, July 2003.

[4] A. H. Sayed, *Fundamentals of Adaptive Filtering,* IEEE Press, 2003.

[5] A. C. Thompson, *Techniques in Single-Bit Digital Filtering,* PhD Dissertation, RMIT University, Melbourne, 2004.

[6] A. Z. Sadik and Z. M. Hussain, "An Adaptive Algorithm for Ternary Filtering," *IEE Electronics Letters*, vol. 42, issue 7, March 2006.

[7] C. Dick and F. Harris, "FPGA signal processing using sigma-delta modulation," *IEEE Signal Processing Magazine*, pp. 20-35, Jan. 2000.

[8] M. A. Aldajani and A. H. Sayed, "Stability and performance analysis of an adaptive sigma-delta modulator," *IEEE Transactions on Circuits and Systems-II:Analog and Digital Signal Processing*, vol. 48, no. 3, March 2001.

[9] C. M. Zierhofer, "Adaptive sigma-delta modulatoion with one-bit quantization," *IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing*, vol. 47, no. 5, March 2000.

[10] H. J. Butterweck, "A wave theory of long adaptive filters," *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 48, no. 6, June 2001.

[11] G. I. Bourdopoulos, "Adaptive order reduction scheme for high-order single-bit $\Delta\Sigma$ modulators," *IEEE Transactions on Circuits and Systems-II: Express Briefs*, vol. 51, no. 5, May 2004.

[12] G. A. Clark, S. K. Mitra, and S. R. Parker, "Block implementation of adaptive digital filter," *IEEE Transactions on Circuits and Systems*, vol. CAS-28, no. 6, June 1981.

[13] J. J. Shynk, "Frequency-domain and multirate adaptive filtering," *IEEE Signal Processing Magazine*, pp. 15-37, Jan. 1992.

[14] B. Widrow *et al*, "Adaptive noise cancelling: principles and applications," *Proceedings of the IEEE*, vol. 63, no. 12, Dec. 1975.

[15] J. G. Proakis *et al*, *Contemporary Communication Systems Using Matlab,* 2nd Edition, Brooks Cole: Pacific Grove, 2003.