Filtering with observation in a manifold: Theory and solution methods

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Outline

Introduction

The classical problem
  Problem statement
  Mathematical formulation
  Solution methods

Diffusions on manifolds
  Brownian Motion
  general case
  Manifolds in s.p.

Observation in a manifold
  Problem statement
  KS and Zakai
  Particle filter

Bibliography
Goals of Presentation

- Classical stochastic filtering: (a.g.w.n.)
  - Theory and solution methods
  - Analogy with other fields
- Diffusions on manifolds (Why?)
  - How to simulate them?
  - S.P. applications
- Observation in a manifold
  - Generalised filtering equation
  - Extension of the particle method

Said & Manton.
Filtering with observation in a manifold: Reduction to a Classical filtering problem.
SIAM Control and Optimisation, 51, 767, 2013.
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- Classical stochastic filtering: *(a.g.w.n.)*
  - Theory and solution methods
  - Analogy with other fields

- Diffusions on manifolds *(Why?)*
  - How to simulate them?
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Problem statement

estimate the state of a system from noisy observation

- System $x$ a Markov process (finite state, diffusion, etc)
- Observation $y$ (real valued)

$$y_t = \int_0^t h(x_s)ds + w_t$$

$h$ sensor function, $w$ Brownian motion independent from $x$

additive Gaussian white noise observation model

Problem: find the conditional distribution $\pi_t$ of $x_t$ given $\mathcal{Y}_t$

$$\int_x \varphi(x)\pi_t(x) = \mathbb{E} [\varphi(x_t)|\mathcal{Y}_t]$$
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x is a (hidden) time homogeneous Markov process

Kolmogorov forward equation $p_t$ "density" of $x_t$

$$p_{t+\delta}(x) = \int_{x'} p_t(x') q_\delta(x', x)$$

$$\partial_t p_t(x) = A^* p_t(x)$$

diffusion approximately,

$$q_\delta(x', x) = \mathcal{N}(x' + \mu(x')\delta, \sigma^2(x')\delta)$$

$$A^* = -\frac{d}{dx} \mu(x) + \frac{1}{2} \frac{d^2}{dx^2} \sigma^2(x)$$
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Abstract Bayes formula

discretise observation

\[ \delta y_t = h(x_t)\delta + \delta w_t \quad \ell(x_t; \delta y_t) = \exp[h(x_t)\delta y_t - (1/2)h^2(x_t)\delta] \]

approximate denormalised conditional density

\[ \rho_t(x) = \int_{x_{n-1}} \ell(x_{n-1})q_\delta(x, x_{n-1}) \cdots \int_{x_0} \ell(x_0)p_0(x_0)q_\delta(x_0, x_1) \]

\[ y \text{ is considered fixed! (}\tilde{x} \text{ independent copy)} \]

\[ \int_x \varphi(x)\rho_t(x) = \mathbb{E}\left[ \varphi(\tilde{x}_t)\prod_{i=0}^{n-1} \ell(\tilde{x}_i; \delta y_i) \right] \]

let \( \delta \downarrow 0 \Rightarrow \) KS formula

\[ \int_x \varphi(x)\rho_t(x) = \mathbb{E}\left[ \varphi(\tilde{x}_t)\exp\left( \int_0^t h(\tilde{x}_s)dy_s - (1/2) \int_0^t h^2(\tilde{x}_s)ds \right) \right] \]
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Zakai equation

stochastic pde for denormalised density

first order approximation

\[ \rho_{t+\delta}(x) \approx \int_{x'} \ell(x', \delta y_t) \rho_t(x') q_\delta(x', x) \]

\[ \rho_{t+\delta}(x) \approx \int_{x'} \rho_t(x') q_\delta(x', x) + \left( \int_{x'} h(x') \rho_t(x') q_{\delta=0}(x', x) \right) \delta y_t \]

first order expansion

\[ \rho_{t+\delta}(x) - \rho_t(x) \approx A^* \rho_t(x) \delta + \rho_t(x) h(x) \delta y_t \]

Zakai equation (\( \dot{y}_t \) white noise)

\[ \partial_t \rho_t(x) = A^* \rho_t(x) + \rho_t(x) h(x) \dot{y}_t \]
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Particle system

Monte Carlo implementation of KS formula

$$\int_x \varphi(x) \rho_t(x) = \mathbb{E}[\varphi(\tilde{x}_t) L_t]$$

Particies: $N$ independent copies of $\tilde{x}$

$$\int_x \varphi(x) \hat{\rho}_t(x) = \frac{1}{N} \sum_{n=1}^{N} \varphi(\tilde{x}^n_t) L(\tilde{x}^n; t)$$

- **Resampling**: reduce variance of particle weights

Del Moral.
Particle system

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Chaos expansion

projection of Zakai equation on an orthonormal basis

random distribution \((w_j \text{ i.i.d. standard normal})\)

\[ w_t = \sum_{j \geq 0} w_j \psi_j(t) \quad \{\psi_j\} \text{ o.n.b. in } L^2(0, T) \]

Chaos space \(H_n\) span of \((h_\alpha \text{ Hermite polynomial})\)

\[ h_{\alpha_1}(w_{j_1}) \cdots h_{\alpha_m}(w_{j_m}) \quad \alpha_1 + \ldots + \alpha_m = n \]

Wiener expansion \(L^2(w) = \bigoplus_n H_n\)

Holden et al.  
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A kind of CLT-I

Riemannian manifold $M$ with metric $\langle \cdot, \cdot \rangle$

Geodesic "zero acceleration" curve

$$\gamma(t) \quad \gamma(0) = p \quad \dot{\gamma}(0) = v$$

Hessian tensor $f \in C^2(M)$

$$(f \circ \gamma)(t) = (f \circ \gamma)(0) + \langle \nabla f, v \rangle t + (1/2) \nabla^2 f(v, v) t^2 + o(t^2)$$

Laplacian trace of Hessian

$$\Delta f(p) = \sum_i \nabla^2 f(e_i, e_i) \quad \{e_i\} \text{ tangent o.n.b. at } p$$
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A kind of CLT-II

successive random geodesics due to “impulse"

Random geodesic

\[ \nu = \sum w^i e_i \text{ and } w^i \text{ white, unit variance} \]

Effect on \( f \)

\[ \mathbb{E}(f \circ \gamma)(\delta) = \mathbb{E}(f \circ \gamma)(0) + (\delta/2)\Delta f(p) + o(\delta^2) \]

Scaling independent steps \( \Rightarrow \) process \( y \)

\[ \mathbb{E}f(y_t) = \mathbb{E}f(y_0) + (1/2) \int_0^t \mathbb{E}\Delta f(y_s) ds \]
A kind of CLT-II

successive random geodesics due to “impulse"

Random geodesic

\[ \nu = \sum w^i e_i \] and \( w^i \) white, unit variance

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Filtering with observation in a manifold
Diffusions on manifolds
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Itô line integral

integral using geodesic increments

Tentative form of integral

$$\int_0^t \langle E, dy_s \rangle = \lim_{\delta \downarrow 0} \sum_{k\delta < t} \langle E_{k\delta}, I(y_{k\delta}, y_{(k+1)\delta}) \rangle$$

In order to have good properties (independence of increments)

$I : M \times M \rightarrow TM$ geodesic interpolation rule

Generalised Itô formula

$$df(y_t) = \langle \nabla f, dy_t \rangle + \frac{1}{2} \Delta f(y_t) dt$$
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elliptic diffusion

elliptic diffusion: can be written as b.m. with drift

Generator

\[ A = H + \frac{1}{2} \Delta \]

\( H \) vector field

Simulation

can also be carried out using geodesics

Application stochastic gradient

\[ A = -\nabla F + \frac{\sigma^2}{2} \Delta \implies \text{stationary density } \propto e^{-2F/\sigma^2} \]
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S.P. Applications

A vision towards stochastic algorithms

classical matrix manifolds (symmetric spaces)
  - $P_+(n)$ p.d.s. matrix ⇒ covariance matrix
  - $V_{n,k}$ Stiefel manifold ⇒ sensor arrays
  - $G_{n,k}$ Grassmann manifold ⇒ subspace

Algorithms
  - Riemannian center of mass ⇒ model change detection
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  Problem statement
  Mathematical formulation
  Solution methods

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A model problem

Estimate angular velocity $\omega$ from partial observation of pose $y$

State model $\omega \in \mathbb{R}^3$ unknown constant
Observation model $y$ evolves on unit sphere

$$\dot{y}_t = -y_t \times \{\omega \, dt + w_t\} \quad \text{w_t vector white noise}$$

Comment on model Rigid body mechanics
Innovation structure what plays the role of $\delta y_t$

- Noiseless case
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Problem statement

Riemannian manifold $M$ with metric $\langle \cdot, \cdot \rangle$

SDE common appearance

\[
\dot{y}_t = H(y_t, x_t) + \sum_r (Y_t) w_t^r
\]

Generator (conditionally on $x$)

\[
A = H_x + \frac{1}{2} \Delta
\]

Intuition

\[
\delta y_t \in T_{y_t}M \quad \delta y_t = H(x_t, y_t)\delta + \delta w_t^i e_i
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Interpolation rules

Interpolation rule

$$\delta y_t = I(y_t, y_{t+\delta}) \quad I : M \times M \rightarrow TM$$

How to choose $I$?

$$\ell(x_t; \delta y_t) = \exp \left[ \langle H_t, \delta y_t \rangle - \frac{1}{2} |H_t|^2 \delta \right]$$

A variety of choices

- Geodesic interpolation
- Second order equivalence

Stochastic integral

$$\int_0^t \langle E, dy_s \rangle = \lim_{\delta \downarrow 0} \sum_{k\delta < t} \langle E_{k\delta}, I(y_{k\delta}, y_{(k+1)\delta}) \rangle$$
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KS and Zakai equation

Likelihood process

\[ L_t = \exp \left( \int_0^t \langle H, dy_s \rangle - \frac{1}{2} \int_0^t |H_s|^2 \, ds \right) \]

KS formula

\[ \int_x \varphi(x) \pi_t(x) = \frac{\int_x \varphi(x) \rho_t(x)}{\int_x \rho_t(x)} \quad \int_x \varphi(x) \rho_t(x) = \mathbb{E} [\varphi(\tilde{x}_t)L_t] \]

Zakai equation

\[ \partial_t \rho_t(x) = A^* \rho_t(x) + \rho_t(x) \langle H, \dot{y}_t \rangle \]
Particle filter

Exactly the same approach as in the classical case

\[
\int_x \varphi(x) \hat{\rho}_t(x) = \frac{1}{N} \sum_{n=1}^{N} \varphi(\tilde{x}_t^n) L(\tilde{x}_t^n; t)
\]

Back to model problem
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