# ANALOG ENVELOPE CONSTRAINED FILTER WITH INPUT UNCERTAINTY

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**ABSTRACT**

In an envelope constrained filtering problem with uncertain input (ECUI), we require the response of the filter to each signal in the input mask to lie within a prescribed output mask. The objective is to design the filter so as to minimize the noise enhancement whilst satisfying these constraints. The continuous-time ECUI problem was previously solved using filter structures which in practice were not physically realisable. This paper presents a new sub-optimal method which allows the use of more realistic filters, and which can converge to the optimal result provided a certain parameter is made small.

## 1. INTRODUCTION

In signal processing the design of many filters can often be cast as a constrained optimization problem where the constraints are defined by the specifications of the filter. These specifications can arise either from practical considerations or from the standards set by certain regulatory bodies. For example, in telecommunication systems, pulse shapes used in transmission systems [7] are specified using templates by recommendations issued by standards bodies (see e.g. [1] and [2]).

The continuous-time envelope-constrained (EC) filtering problem considers the design of a filter such that the noiseless response $\psi$ to a specified excitation $s$ fits into an envelope described by $\varepsilon^+$ and $\varepsilon^-$, as shown in Figure 1. The EC with uncertain input (ECUI) filtering problem addresses the robustness to input disturbances by allowing for uncertainty in the input pulse. Here the input is not specified exactly, but is known to lie within an input envelope described by upper and lower boundaries $s^+$ and $s^-$. The filter is required to fit the response of all excitations within the boundaries $s^+$ and $s^-$ into the output mask.

The discrete-time ECUI problem was first addressed for FIR filters in [3]. In [10], the continuous-time ECUI problem was formulated as a quadratic program with non-differentiable constraints. This problem was then solved by transforming it into a positive definite QP problem with affine (hence differentiable) constraints. For finite-dimensional filters, this transformation requires their impulse responses to possess very restrictive properties that are not physically realisable with analog or hybrid components [11].

In this paper, we present a new method for solving the ECUI problem by approximating the non-differentiable constraints with differentiable ones. It can be shown that if a solution satisfies these differentiable constraints, then it will also satisfy the original non-differentiable constraints. The advantage of this approach is that the approximation assumes no specific property on the filter impulse reponse, thus allowing solutions to the ECUI problem for filter structures which are physically realisable. Numerical simulations are given for both an analog filter, and a hybrid filter which consists of both digital and analog components.

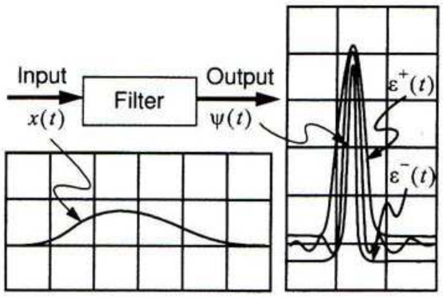


**Fig. 1**. EC filter

## 2. PROBLEM STATEMENT

This section presents a general formulation of the ECUI filtering problem. The reader is referred to [10], [11] and [12] for derivations and further details.

Here $x$ denotes the input, $u$ denotes the impulse response of the filter, $d \equiv 0.5(\varepsilon^+ + \varepsilon^-)$ denotes the desired output, $\varepsilon \equiv 0.5(\varepsilon^+ - \varepsilon^-)$ denotes allowable deviation from the desired output, $s \equiv 0.5(s^+ + s^-)$ denotes the nominal input and $\theta \equiv 0.5(s^+ - s^-)$ denotes the uncertainty on the nominal input. The general ECUI filtering problem can be stated as the following optimization problem on a Hilbert space:

$$\min_{u \in H} f(u) \quad = \quad \langle u, Lu \rangle, \tag{1}$$

$$\text{subject to } |\Xi_x \Psi u - d| \quad \preceq \quad \varepsilon, \forall x : |x - s| \preceq \theta \tag{2}$$

where $L : H \to H$, $\Psi : H \to Y$ and $\Xi_x : Y \to C$ are linear operators, $H$ is the Hilbert space of filter impulse responses, $Y$ is some vector space and $C$ is the space of filter outputs. The partial ordering $\preceq$ on $C$ is defined for all $x, y \in C$ by: $x \preceq y$ if and only if $x(t) \leq y(t)$ for all $t \in \Omega$. $d$ and $\varepsilon$ are assumed to be bounded, and $\theta$ and $s$ are finite energy signals with $\theta \succeq 0$.

In this paper we consider 2 types of filters: 1) analog filters and 2) hybrid filters, which comprise both digital and analog components as shown in Figure 2. The ECUI problems for analog and hybrid filters are both special cases of problem (1-2).

For analog filters, $H = L_2(\mathbf{R}_+)$, the space of square integrable functions on $\mathbf{R}_+ \equiv [0, \infty)$, with inner product $\langle x, y \rangle =$

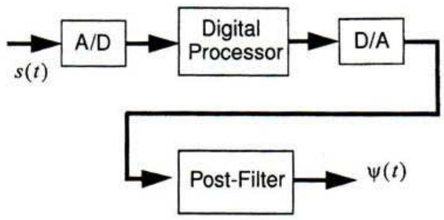


**Fig. 2**. Block diagram of hybrid filter

$\int x(t)y(t)dt$, $Y = H$ and $C = L_\infty(\mathbf{R}_+)$ is the Banach space of essentially bounded functions.
$L{:}L_2(\mathbf{R}_+)\to L_2(\mathbf{R}_+)$ is a non-negative self-adjoint linear operator defined by

$$(Lu)(t) = \int R_{nn}(t-\lambda)u(\lambda)d\lambda$$

with $R_{nn}$ being the autocorrelation of an additive, zero-mean and stationary input noise process, $\Psi$ is the identity operator and $\Xi_x$ is the convolution operator, i.e.

$$(\Xi_x u)(t) = (x*u)(t) = \int x(t-\lambda)u(\lambda)d\lambda$$

For hybrid filters, $H = l_2$, the space of square summable sequences, with inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x(i)y(i)$, $Y = C = L_\infty(\mathbf{R}_+)$. Let $T_s$ denote the sampling period of the A/D converter and $\Lambda$ denote the response of the post-filter to a rectangular pulse of length $T_s$. Then, the positive semi-definite linear operator $L$ is defined by

$$(Lu)(l) = \sum_{m=1}^{\infty} u(m)\Gamma_{l-m}$$

where

$$\begin{aligned}\Gamma_{l-m} &= \frac{1}{T_s}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} R_{nn}((j-k)T_s) \\ &\quad\times \int_0^{T_s} \Lambda[t-(l-m+j)T_s]\Lambda[t-kT_s]dt\end{aligned}$$

$\Psi$ and $\Xi_x$ are defined by

$$\begin{aligned}(\Psi u)(t) &= \sum_{j=0}^{\infty} \Lambda(t-jT_s)u(j), \\ (\Xi_x v)(t) &= \sum_{k=-\infty}^{\infty} v(t-kT_s)x(kT_s)\end{aligned}$$

## 3. PROBLEM TRANSFORMATION

The problem statement in section II is not useful for computational purposes since to determine the feasibility of a filter, one would need to compute its response to every signal in the input mask. There does not seem to be any standard numerical techniques for handling problems with constraints of this form.

### 3.1. Constraint Transformation

We would like to have an equivalent but more explicit expression which does not contain the perturbed input $x$, but involves only $d$, $\varepsilon$, $s$, $\theta$ and the filter's impulse response $u$. In [10], it has been shown that the general ECUI filtering problem is equivalent to:

$$\min_{u\in H} f(u) = \langle u, Lu\rangle,$$
$$\text{subject to } |\Xi_s\Psi u - d| + \Xi_\theta|\Psi u| \preceq \varepsilon$$

The feasible region can thus be written as

$$\mathcal{F} = \{u\in H : |\Xi_s\Psi u - d| + \Xi_\theta|\Psi u| \preceq \varepsilon\}$$

### 3.2. Approximation By A Smooth Problem

The transformed problem as stated is generally a non-smooth optimization problem because the constraint function

$$G(u) = |\Xi_s\Psi u - d| + \Xi_\theta|\Psi u| - \varepsilon$$

is not everywhere differentiable with respect to the variable $u$. The first absolute value operator in $G$ does not pose any problem since it is straight forward to show that

$$|\Xi_s\Psi u - d| + \Xi_\theta|\Psi u| - \varepsilon \preceq 0 \Leftrightarrow \begin{cases} \Xi_s\Psi u + \Xi_\theta|\Psi u| - \varepsilon^+ \preceq 0 \\ -\Xi_s\Psi u + \Xi_\theta|\Psi u| + \varepsilon^- \preceq 0 \end{cases}$$

The major difficulty is caused by the term $\Xi_\theta|\Psi u|$. In this paper, we will present a method to modify the constraints by replacing the absolute value operator with another operator that closely approximates it, but which is differentiable everywhere. The idea is to replace the absolute value function with a quadratic, in the neighbourhood of points where the function is non-differentiable, ie. at zero. We can define such a function $|\bullet|_\triangle : \mathbf{R} \to \mathbf{R}$, where $\Delta > 0$, as follows:

$$|x|_\triangle = \begin{cases} |x| & ,|x| \geq \Delta \\ \frac{1}{2\Delta}x^2 + \frac{\Delta}{2} & ,|x| < \Delta \end{cases}$$

This function is differentiable everywhere. The 'smoothed' ECUI filtering problem can thus be stated as:

$$\min_{u\in H} f(u) = \langle u, Lu\rangle,$$
$$\text{subject to } |\Xi_s\Psi u - d| + \Xi_\theta|\Psi u|_\triangle \preceq \varepsilon$$

Let $u^0$ and $u^0_\triangle$ be the optimal solutions to the non-smooth and smooth ECUI problems respectively. Intuitively, as $\Delta$ approaches zero, we have a progressively better approximation to the absolute value function and hence $u^0_\triangle$ can approximate $u^0$ to any degree of accuracy. Formally it can be shown that $u^0_\triangle$ also satisfies the non-smooth constraints. Furthermore, as $\Delta \to 0$, $f(u^0_\triangle) \to f(u^0)$ and $\lim_{\Delta\to 0} ||u^0 - u^0_\triangle|| = 0$.

## 4. FINITE DIMENSIONAL APPROXIMATION

The optimum ECUI filter in an infinite dimensional Hilbert space $H$ is only of theoretical interest since it is unlikely that these filters can be realised with finite circuitry without violating the constraints. It would seem more appropriate to choose a particular finite-dimensional filter structure and then impose the constraints. This section addresses the issue of finite dimensional approximation.

Let the finite-structured filters have impulse responses given by $u = K\boldsymbol{a} = \sum_{i=0}^{n-1} a_i v_i$, where $\{v_i\}_{i=0}^{\infty}$ is a total sequence in $H$ and $K : \mathbf{R}^n \to H$ is a bounded linear operator. The feasible region $\mathcal{F} \cap [\{v_i\}_{i=0}^{n-1}]$ ( $[\{v_i\}_{i=0}^{n-1}]$ denotes the span of $\{v_i\}_{i=0}^{n-1}$) is thus embedded in an $n$-dimensional subspace of $H$ and can be characterized by the set of feasible filter coefficients $\mathcal{F}^n = \{\boldsymbol{a} \in \mathbf{R}^n : K\boldsymbol{a} \in \mathcal{F}\}$. The cost functional can be expressed in terms of the vector of filter coefficients $\boldsymbol{a}$, as follows

$$f \circ K(\boldsymbol{a}) = \langle K\boldsymbol{a}, LK\boldsymbol{a} \rangle = \boldsymbol{a}^T K^\dagger LK\boldsymbol{a}$$

where $K^\dagger$ is the adjoint operator of $K$. The operator $K^\dagger LK : \mathbf{R}^n \to \mathbf{R}^n$ can be represented by the following $n \times n$ matrix (often called a Gram matrix)

$$\begin{bmatrix} \langle v_0, Lv_0 \rangle & \langle v_0, Lv_1 \rangle & \dots & \langle v_0, Lv_{n-1} \rangle \\ \langle v_1, Lv_0 \rangle & \langle v_1, Lv_1 \rangle & \dots & \langle v_1, Lv_{n-1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_{n-1}, Lv_0 \rangle & \langle v_{n-1}, Lv_1 \rangle & \dots & \langle v_{n-1}, Lv_{n-1} \rangle \end{bmatrix}$$

By increasing the number of elements used from any total sequence in $H$, it can be shown that the solution will converge to the optimal solution (see [10]), hence we can approximate the optimal solution to any degree of accuracy, whilst each approximant still satisfies the constraints. With an appropriate choice of basis these sub-optimal solutions correspond to finite-structured filters which can be readily realised. The finite dimensional ECUI problem can be stated as

$$\min_{\boldsymbol{a} \in \mathbf{R}^n} \boldsymbol{a}^T K^\dagger LK\boldsymbol{a},$$
$$\text{subject to } |\Xi_s \Psi K\boldsymbol{a} - d| + \Xi_\theta |\Psi K\boldsymbol{a}| \preceq \varepsilon$$

By replacing $|\bullet|$ with $|\bullet|_\triangle$ we obtain the following smoothed approximation

$$\min_{\boldsymbol{a} \in \mathbf{R}^n} \boldsymbol{a}^T K^\dagger LK\boldsymbol{a},$$
$$\text{subject to } |\Xi_s \Psi K\boldsymbol{a} - d| + \Xi_\theta |\Psi K\boldsymbol{a}|_\triangle \preceq \varepsilon$$

## 5. EXAMPLES

This section presents numerical results for 1) an analog Laguerre filter (see [6], [8] ), and 2) a hybrid filter which uses an FIR filter as the digital processor (see [11]).

### 5.1. Example using Laguerre functions

In most practical analog realisations, $L$ is the identity operator and $K$ is specified by a finite subset of a complete set of orthonormal $v_i$'s. In this case it is easily seen that the Gram matrix is the identity matrix. Suppose the input noise $\xi$ is also white, i.e. the auto-correlation $R_{nn}(\tau) = N_0 \delta(\tau)$, where $\delta$ is a unit impulse. Then the output noise power, or cost functional, is proportional to $N_0 \|u\|_2^2$.

The Laguerre polynomials $L_k(t)$ are defined (see [4]) for each integer $k \geq 0$ as

$$L_k(t) = \sum_{i=0}^{k} \binom{k}{k-i} \frac{(-t)^i}{i!}$$

The basis functions used here are given by

$$v_k^p(t) = \sqrt{2p} e^{-pt} L_k(2pt)$$

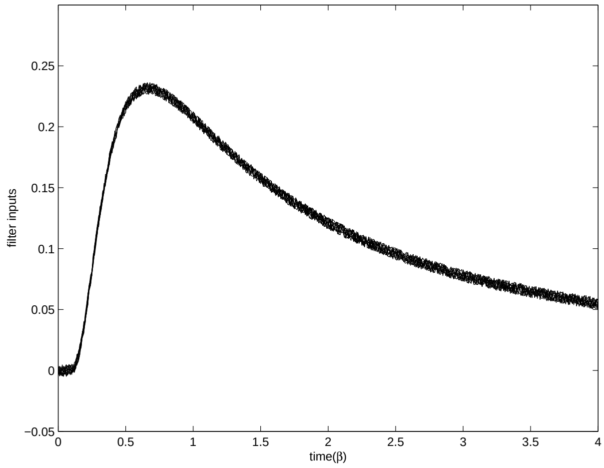


**Fig. 3**. Laguerre filter example - inputs

where $p > 0$ and $k$ is a non-negative integer. The sequence $\{v_k^p\}_{k=0}^{\infty}$ forms an orthonormal basis for $L_2(\mathbf{R}_+)$, and is also known as the Laguerre basis. Filter impulse responses of the form $\sum_{i=0}^{n-1} a_i v_i^p$ can be readily realised with passive analog circuitry.

This example concerns the equalization of a digital transmission channel consisting of a coaxial cable operating at the DSX3 rate (45Mb/s) ([1] , [2]). The coaxial cable has a 20dB attenuation at a normalized frequency of $2\pi/\beta$, where the baud interval $\beta = 22.35 \times 10^{-9}(s)$. The input noise is assumed to be white. An equalizing filter is required to shape all signals within a 2% tolerance of the cable's impulse response so that they fit in the envelope given by the DSX3 pulse template (see Figure 4).

We use $p = 12$, $k = 8$ and $\Delta = 0.01$. The effective duration of the input is taken to be $20\beta$. For computational purposes, the continuum of constraints is discretized at 100 constraints per Baud interval. This results in a constrained optimization problem with a finite number of inequality constraints, which can be solved using MATLAB. Figure 3 shows some signals which were randomly perturbed about the nominal input but lie within the input mask. Figure 4 shows the filter's response to these signals. We can see that the responses all stay within the output mask. The value of the cost function is 75.92.

### 5.2. Example using an FIR digital processor

For hybrid filters, let $[K^\dagger LK]_{j,k}$ denote the $(j,k)$th entry of the matrix $K^\dagger LK$. Then

$$[K^\dagger LK]_{j,k} = \frac{N_0}{T_s} \sum_{l \in \tilde{\Omega}_u} \sum_{m \in \tilde{\Omega}_u} v_j(l) v_k(m) \times \int_{-\infty}^{\infty} \Lambda(\zeta - lT_s) \Lambda(\zeta - mT_s) d\zeta$$

If the digital processor is an FIR filter, the linear operator $K$ becomes the identity operator and

$$[K^\dagger LK]_{j,k} = \Gamma_{j-k} = \frac{N_0}{T_s} \int_{-\infty}^{\infty} \Lambda[\zeta - (j-k)T_s] \Lambda(\zeta) d\zeta$$

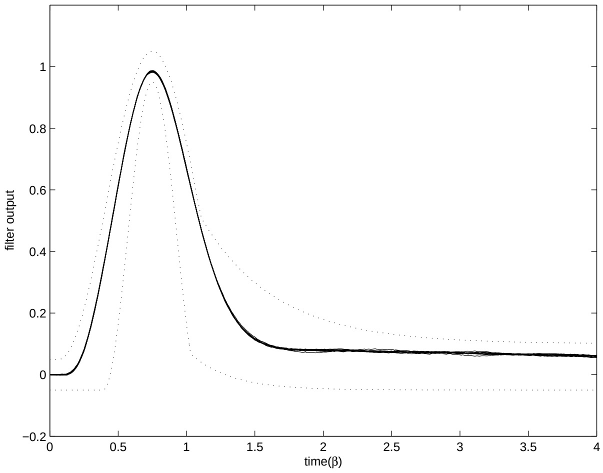


**Fig. 4**. Laguerre filter example - outputs

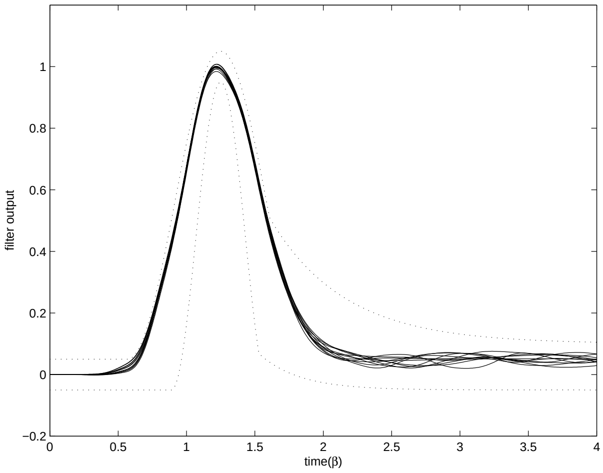


**Fig. 5**. Hybrid filter example - outputs

The impulse response $u$ is given by

$$u(k) = a_k, k = 0, 1, \ldots, n-1$$

This example again concerns the equalization of a digital transmission channel consisting of a coaxial cable operating at the DSX3 rate. The same nominal input is used, except the perturbed inputs are now sampled at every $T_s = \beta/4$. We use a 20-tap FIR filter, and a smoothing parameter $\Delta = 0.01$. For computational purposes, the continuum of constraints is discretized at 100 constraints per Baud interval. The post-filter is a 5th order Butterworth low-pass filter with cut-off frequency equal to half the sampling frequency. Figure 5 shows the filter's response to perturbed input signals, and we can see that all the responses fit inside the mask. The value of the cost function is 18.04.

## 6. CONCLUSIONS

The envelope constrained filtering problem with uncertain input for analog and hybrid filters can be formulated as a non-smooth semi-infinite programming problem. The non-differentiable constraints presents a barrier for applying gradient based approaches for solving this problem. In this paper, we have proposed a smooth approximation to the non-differentiable constraints. This enables the envelope constrained filtering problem with uncertain input to be solved for realisable filter structures. This technique has been sucessfully applied to the design of analog and hybrid filters in a communication channel equalization application.

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